Project *MATHEMATICS!*

Program Guide and Workbook

to accompany the videotape on

THE THEOREM OF PYTHAGORAS

Project MATHEMATICS!
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The videotape on

THE THEOREM OF PYTHAGORAS

was funded by

ACM/SIGGRAPH, the Association of Computing Machinery's Special Interest Group on Computer Graphics

and

THE EDUCATIONAL FOUNDATION OF AMERICA,

and was produced by

PROJECT MATHEMATICA,

now known as

MATHEMATICS !

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AIMS AND GOALS OF *MATHEMATICS !*

The primary purpose of *Mathematics !* is to use computer-animated videotapes to show students that learning mathematics can be exciting and intellectually rewarding. The videotapes treat mathematical concepts in ways that cannot be done at the chalkboard or in a textbook. They provide an audiovisual resource to be used together with textbooks and classroom instruction. Each videotape is accompanied by a workbook designed to help instructors integrate the videotape with traditional classroom activities. Video makes it possible to transmit a large amount of information in a relatively short time. Consequently, it is not expected that all students will understand and absorb all the information in one viewing. Video technology permits the viewer to stop the tape and repeat portions as needed.

The manner in which the videotape is used in the classroom will depend on the ability and background of the students and on the extent of teacher involvement. Some students will be able to watch the tape and learn much of the material without the help of an instructor. However, most students cannot learn mathematics by simply watching television any more than they can by simply listening to a classroom lecture or reading a textbook. For them, interaction with a teacher is essential to learning. The videotapes and workbooks are designed to stimulate discussion and encourage such interaction.

STRUCTURE OF THE WORKBOOK

The workbook begins with a brief outline of the contents of the videotape, followed by suggestions of what the teacher can do before showing the tape. The workbook is divided into numbered sections corresponding to capsule subdivisions in the tape. Each section summarizes the important points in the capsule and contains exercises that can be used to strengthen understanding. The exercises emphasize key ideas, words and phrases, as well as applications. Some sections suggest special projects that students can do for themselves.

I. BRIEF OUTLINE OF THE PROGRAM

The videotape begins with a brief computer-animated *Review of Prerequisites* that contains excerpts from earlier programs dealing with concepts the student should be familiar with. The program itself begins with three real-life situations that lead to the same mathematical problem:

How do you find one side of a right triangle if you know the other two sides?

The question is answered by a simple computer-animated derivation of the Pythagorean theorem (based on similar triangles):

In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the two legs.

The algebraic formula $a^2 + b^2 = c^2$ is interpreted geometrically in terms of areas of squares, and is then used to solve the three real-life problems referred to above. Historical context is provided through stills showing Babylonian clay tables and various editions of Euclid's *Elements*. The tape exhibits several different computer-animated proofs of the Theorem of Pythagoras, and extends it to 3-dimensional space. The last capsule of the tape, entitled *Previews of Things to Come*, shows how the Pythagorean theorem is used in trigonometry, and points out that the theorem does not hold for spherical triangles.

II. BEFORE WATCHING THE VIDEOTAPE

The videotape builds on three main ideas the student should be familiar with. These are listed below and are demonstrated in the first capsule of the video in a section entitled *Review of Prerequisites.* If students are familiar with these ideas, this section will serve as a review. If not, an effort should be made to acquaint them with these ideas and with the key words and statements listed below before viewing the tape. A good way to do this is to have the students read the next section and solve the exercises.

KEY WORDS AND STATEMENTS:

Hypotenuse and *legs* of a *right triangle.*

The *area of a parallelogram* is equal to the product of its base and altitude.

The *area of a triangle* is one-half the product of its base and altitude.

Triangles with a common base and equal altitudes are said to be related by *shearing*.

A *change of scale* transforms a figure into a similar one of the same shape but different size.

THE THREE MAIN IDEAS:

Lengths of corresponding sides of similar triangles have the same ratios.

Shearing a triangle does not change its area. Shearing a parallelogram does not change its area.

If the scale for measuring distances is multiplied by a factor r, the area of a figure is multiplied by r^2.

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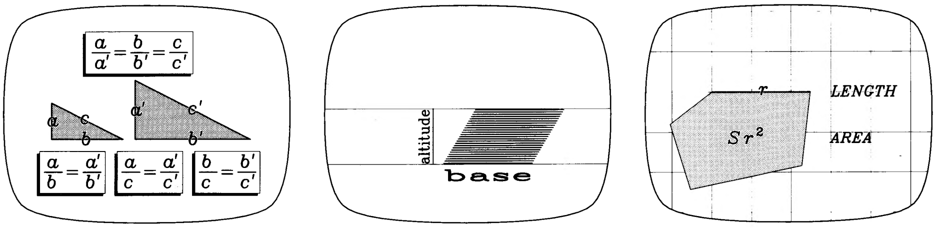

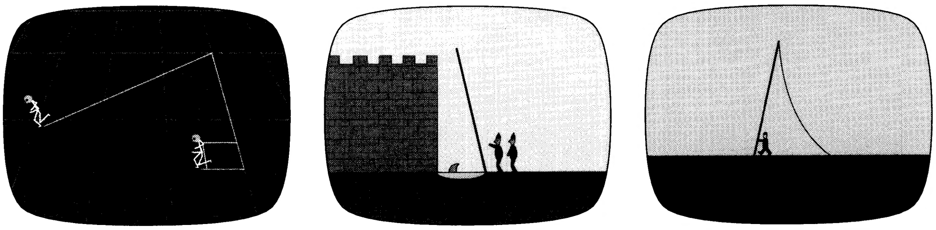

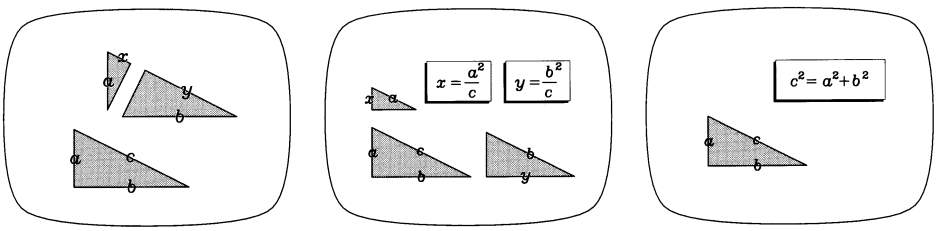

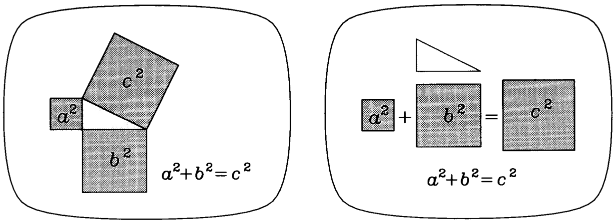

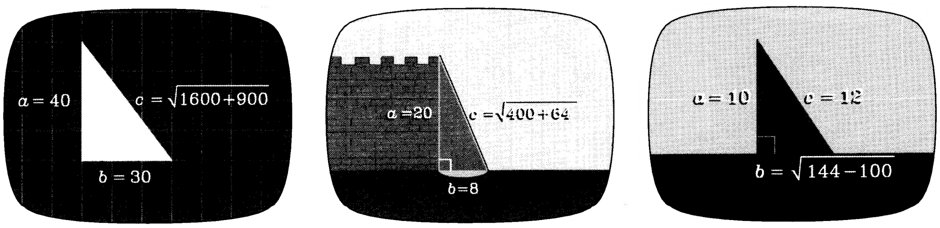

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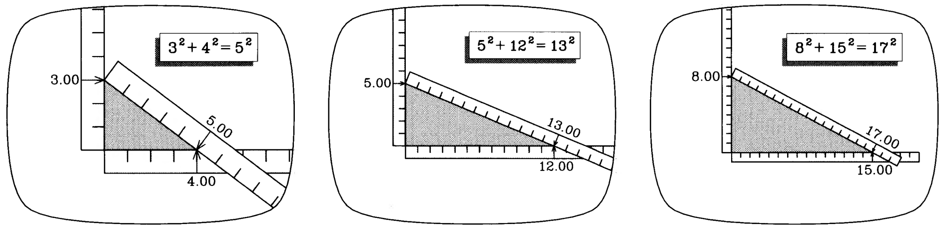

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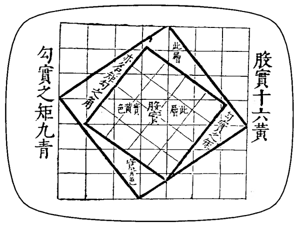

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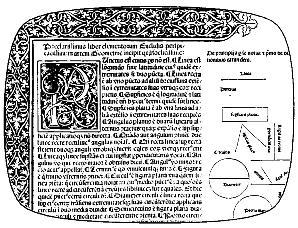

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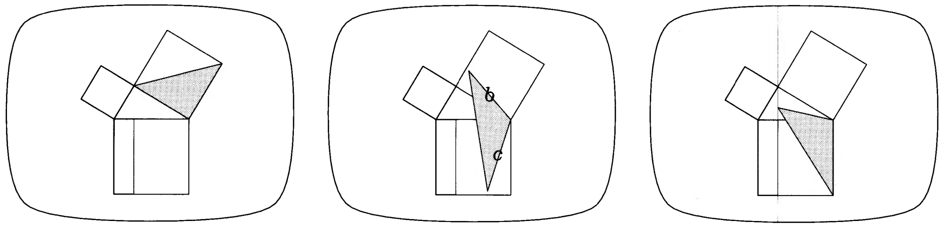

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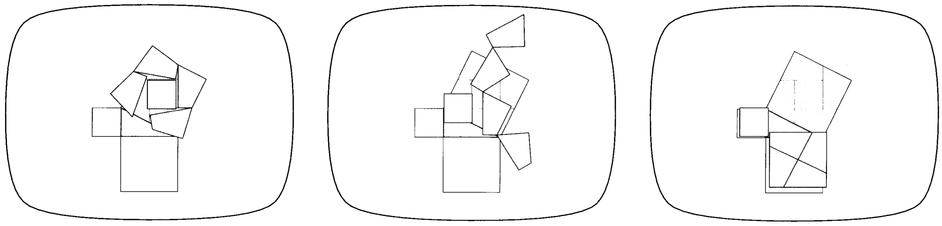

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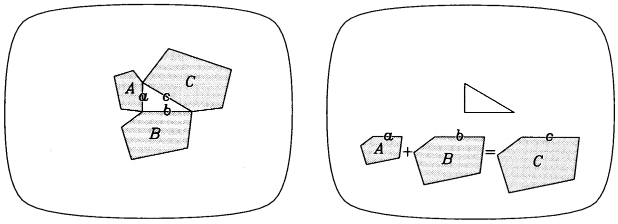

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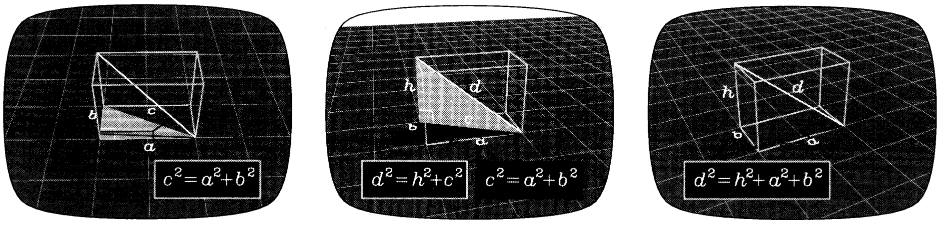


Review of Prerequisites

This section discusses the three main ideas mentioned above.

THE FIRST IDEA: Lengths of corresponding sides of similar triangles have the same ratios.

Similarity of triangles is a simple concept that is used again and again in many applications of geometry. Figure 1 shows two similar triangles. They have the same shape, because corresponding angles are equal, but are of different size, with lengths of corresponding sides being proportional.

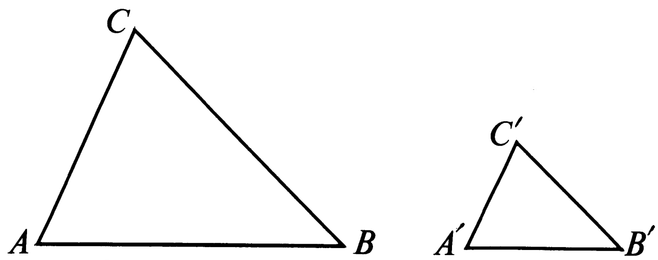


Figure 1. Similar triangles.

In Figure 1, the sides of the larger triangle are twice as long as those of the smaller, so

$$AB = 2A'B', \quad BC = 2B'C', \text{ and } AC = 2A'C' .$$

In general, two triangles ABC and $A'B'C'$ are called *similar* if corresponding angles are equal,

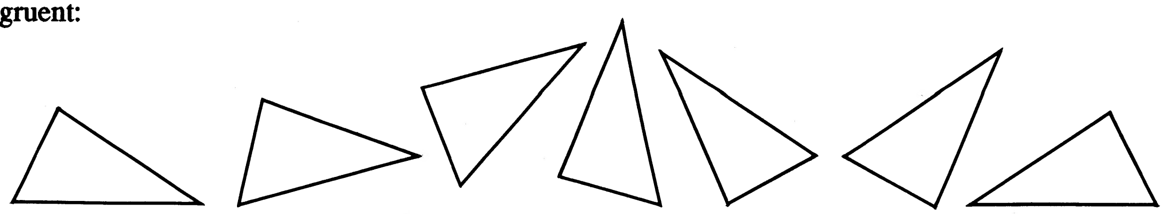
$$\angle A = \angle A', \quad \angle B = \angle B', \quad \angle C = \angle C'.$$

This implies that the lengths of corresponding sides have the same ratio:

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'} .$$

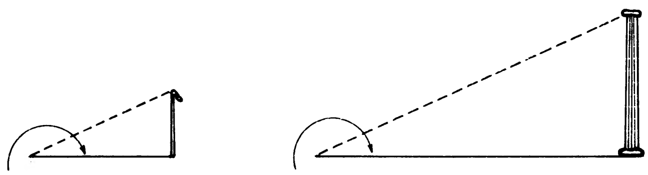
In Figure 1 this common ratio is equal to 2. For arbitrary similar triangles, the common ratio is some positive number r, and we say that triangle ABC is similar to triangle $A'B'C'$ with *similarity ratio* or *scaling factor* r. The number r is called an *expansion factor* if r is greater than 1 (written in symbols as $r > 1$), and a *contraction factor* if r is less than 1 (written $r < 1$). For example, in Figure 1, the lengths of the sides of triangle ABC are twice as long as those of triangle $A'B'C'$, so ABC is obtained from $A'B'C'$ by expansion by the factor $r = 2$. On the other hand, triangle $A'B'C'$ is obtained from ABC by contraction by the factor $r = 1/2$. Expansion or contraction of the sides of a triangle without distortion of the angles always produces a similar triangle.

In summary, similar triangles have the same shape but may be of different size. If they have the same shape and the same size (similarity ratio 1), the triangles are called *congruent*. If a triangle is moved without changing the lengths of its sides, for example if it is shifted, rotated, or flipped over to form a mirror image, a congruent triangle is obtained. All these triangles are congruent:



EXERCISES INVOLVING SIMILAR TRIANGLES

1. The Greek mathematician Thales in the 6th century B.C. is said to have invented a method for determining the height of a column by looking at its shadow and the shadow of his staff. Look at the figure and then explain how his method works.

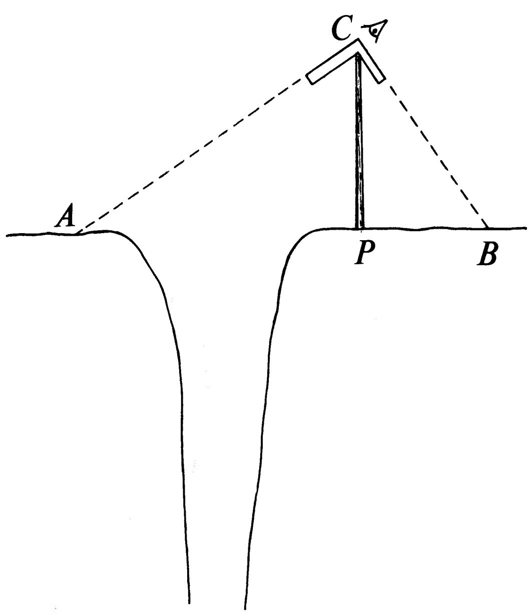


shadow cast by Thales's staff

shadow cast by a column

2. If two triangles ABC and $A'B'C'$ have two of their corresponding angles equal, say $\angle A = \angle A'$ and $\angle B = \angle B'$, show that the third angles are also equal: $\angle C = \angle C'$.

3. A bridge designer needs to know the distance between two points P and A on opposite sides of a deep canyon. He places a vertical rod of length PC on his side of the canyon as shown in the figure. He then holds a carpenter's square with its right angle at C and sights along one leg of the square toward the point A on the opposite side. Without changing the position of the square, he sights along the other leg toward a point B that he marks on the ground. If he knows the lengths PC and PB, explain how he can determine the distance PA.



A carpenter's square used to find the distance to an inaccesible point.

THE SECOND IDEA: Shearing a parallelogram or a triangle doesn't change its area.

The area of a parallelogram is base times altitude. The usual method of proving this is to convert the parallelogram to a rectangle with the same base and altitude by removing a right triangle from one side and replacing it on the other, as indicated in Figure 2. All the parallelograms in Figure 2 have the same base and the same altitude, so they have equal areas.

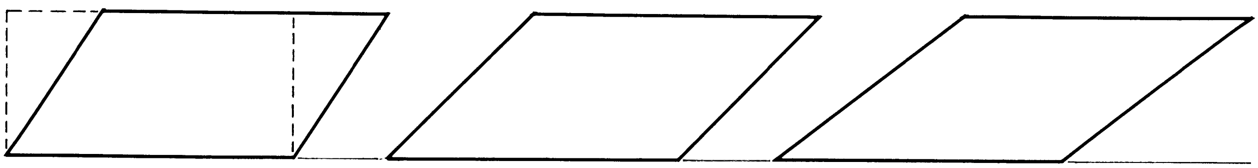


Figure 2. Parallelograms having the same base and altitude also have equal areas.

There is another way to understand why the areas are equal without using the formula for area. Take a deck of cards with one rectangular face painted red, and tip the deck as shown in Figure 3. This process, called *shearing*, transforms the rectangular face of the deck into a parallelogram. The amount of red paint on the faces does not change, so shearing a parallelogram does not change its area. All the parallelograms in Figure 2 are related by shearing, so they all have equal areas. Of course, the deck of cards also illustrates a property of volume. Shearing the deck of cards does not change the volume of the deck. This is true even if the cards are not rectangular.

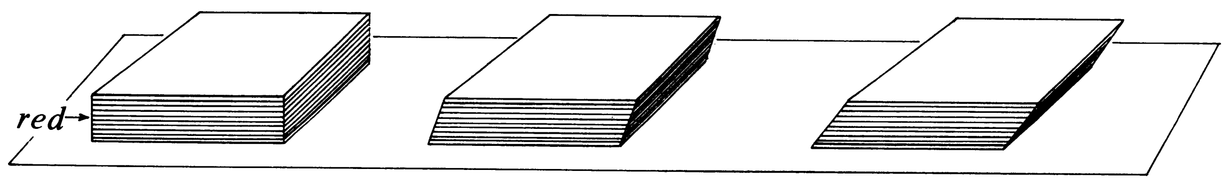


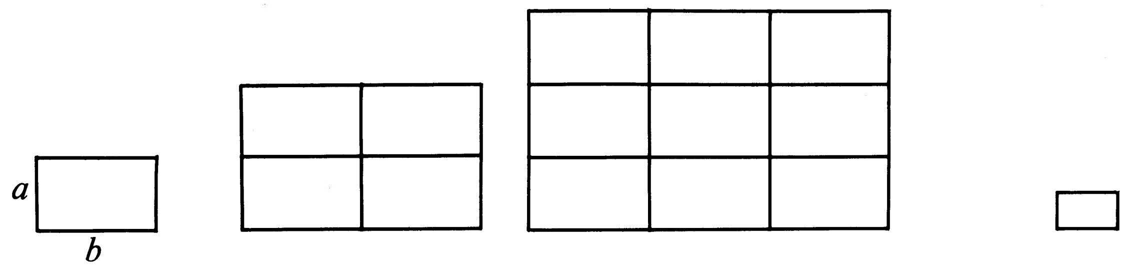
Figure 3. Shearing a deck of cards does not change the area of the red faces, or the volume of the deck.

If each of the red parallelograms in Figure 3 is cut in half as indicated in Figure 4, the red faces are triangles, each with area exactly half that of the parallelogram. Shearing a triangle does not change the amount of red paint covering it, so all the triangles in Figure 4 have equal areas. The videotape uses shearing to illustrate Euclid's proof of the Pythagorean theorem.

Figure 4. Shearing a triangle does not change the area of the triangle.

THE THIRD IDEA: Behavior of area under expansion or contraction.

Take a rectangle with base b and altitude a, as shown in Figure 5. The area of the rectangle is ab, the product of the base and the altitude. Now multiply each side of the rectangle by a factor r. This is called *scaling* the rectangle by a factor r. Scaling produces a new rectangle of sides ra and rb; its area is equal to $(ra)(rb) = r^2ab$. Scaling by various factors is illustrated in Figure 5.



area = ab | scaling factor $r = 2$, area = $4ab$ | scaling factor $r = 3$, area = $9ab$ | scaling factor $r = 1/2$, area = $ab/4$

Figure 5. Scaling a rectangle by a factor r multiplies its area by the factor r^2.

The same is also true for triangles. The area of a triangle is $ab/2$, one-half the base times the altitude, so if each of the base b and altitude a is multiplied by a factor r, the new triangle has area

$$(ra)(rb)/2 = r^2(ab/2).$$

Scaling a triangle by a factor r multiplies its area by the factor r^2.

Now we can show that the same result is true for any polygonal figure. A general polygonal figure can be decomposed into triangular pieces. An example is shown in Figure 6. If we let A denote the area of the polygonal figure and let A_1, A_2, ..., A_5, denote the areas of the triangular pieces, then A is the sum of the areas of the triangular pieces:

$$A = A_1 + A_2 + A_3 + A_4 + A_5.$$

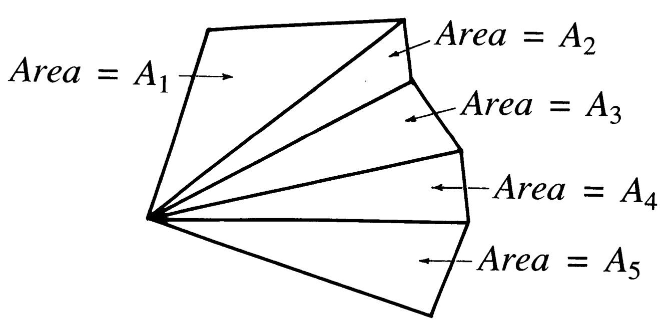


Figure 6. A polygonal figure decomposed into five triangular pieces.

If the polygonal figure is scaled by a factor r, each triangular piece is scaled by the same factor, and the area of each triangular piece gets multiplied by the factor r^2. Thus the scaled polygon has area

$$r^2A_1 + r^2A_2 + r^2A_3 + r^2A_4 + r^2A_5.$$

The common factor r^2 can be factored out and we get

$$\text{area of scaled polygon} = r^2(A_1 + A_2 + A_3 + A_4 + A_5) = r^2A.$$

The same argument works for any polygon. Scaling a polygon by a factor r multiplies its area by the factor r^2.

Now take a more general plane figure F with curved boundaries, and let S denote its area. Scaling by a factor r produces a similar figure F', as shown by the example in Figure 7.

original figure F

area = S

scaled figure F'

area = r^2S

Figure 7. Scaling a plane figure by a factor r multiplies its area by r^2.

The area of F' is equal to r^2S. This is because F can be approximated from the inside and from the outside by polygons, as shown in Figure 8. The area of F lies between the areas of the inner and outer polygons:

$$\text{area of inner polygon} < S < \text{area of outer polygon}.$$

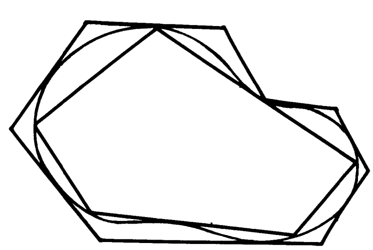


Figure 8. A curved shape approximated by inner and outer polygons.

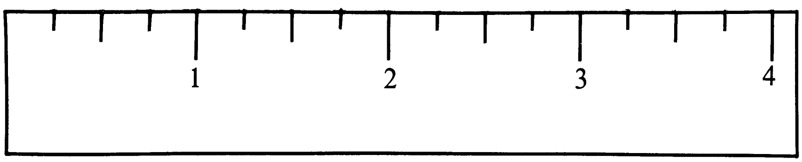
By taking approximating polygons with an increasing number of sides, the areas of the inner and outer polygons can be made arbitrarily close to each other and therefore arbitrarily close to the area of F. If the entire diagram is scaled by a factor r, the approximating polygons are also scaled by a factor r and their areas are multiplied by the factor r^2:

$$r^2(\text{area of inner polygon}) < \text{area of } F' < r^2(\text{area of outer polygon}).$$

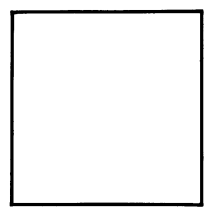
Because the areas of the inner and outer polygons can be made arbitrarily close to S, it follows that the area of F' is equal to r^2S.

APPENDIX: UNIT LENGTH AND UNIT AREA (optional)

Commerce and science share a common need for standardized units of measure. Throughout history, governments and scientific organizations have tried to establish standard units of time, length, area, and weight. This section describes two choices of units for length and area and shows how they are related. For example, Tom chooses the unit length to be one inch (written more briefly as 1 in), as indicated by the ruler shown below. The unit of area is chosen to be a square with edges of unit length. This is called a unit square, and its area is said to be one square inch (written more briefly as 1 in^2). A rectangle with edges a and b measured in inches is said to have area ab square inches.



Ruler based on Tom's unit length, 1 in

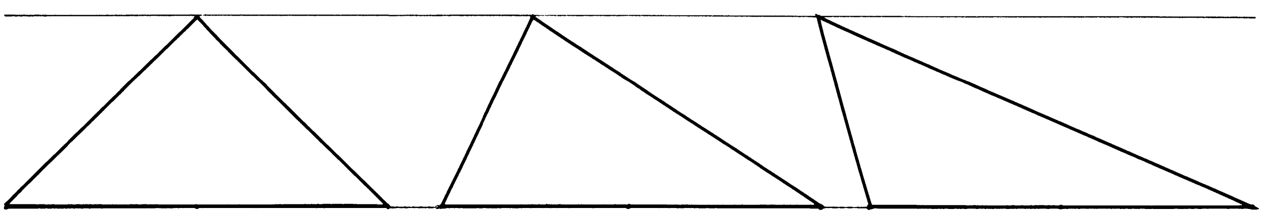


Tom's unit area = 1 in^2

1. Determine the dimensions of this page in Tom's units: width ____ in height ____ in.

   Determine the area of this page in Tom's units: area ___ in^2

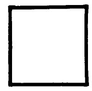
2. Determine the area of each of the following triangles in Tom's units:



(a) area ____in^2 (b) area ____ in^2 (c) area ____in^2

Graph paper is often printed with squares measuring 1 cm (centimeter) on each side, as shown below in Exercise 5. Jane chooses the unit length to be 1 cm, and she chooses the unit square to be one whose edges have unit length. There are 2.54 cm in one inch, so 1 cm is about 0.39 in.

Jane's unit length



Jane's unit area

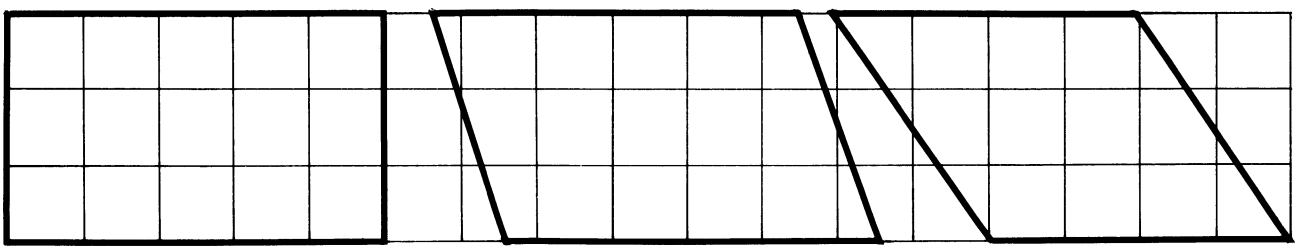
3. Determine the dimensions of this page and its area in Jane's units:

width ____ cm height ____ cm area ____ cm^2

4. Determine the area of each of the triangles in Exercise 2 in Jane's square units:

(a) area ____ cm^2 (b) area ____ cm^2 (c) area ____ cm^2

5. Determine the area of each of the following parallelograms in Jane's square units:



(a) area ____ cm^2 (b) area ____ cm^2 (c) area ____ cm^2

6. Determine the area of each figure in Exercise 5 in Tom's square units.

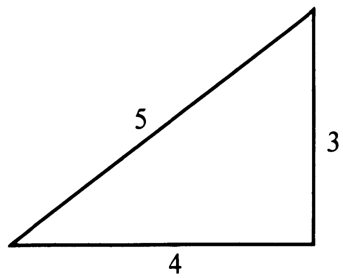
7. If a figure has area T in Tom's square units and area J in Jane's square units, which of the following relations are correct?

(a) $T = (0.39)J$ (b) $J = (0.39)T$ (c) $T = (2.54)J$ (d) $J = (2.54)T$

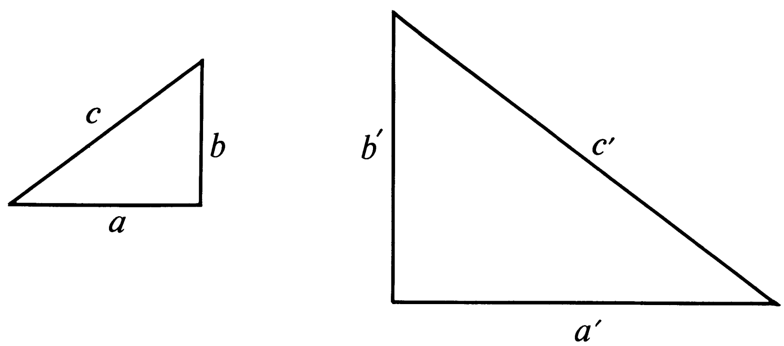
(e) $T = (0.39)^2J$ (f) $J = (0.39)^2T$ (g) $T = (2.54)^2J$ (h) $J = (2.54)^2T$.

8. Based on what you have learned in the foregoing examples, describe a general method for converting the area of a figure from one system of square units to another.

In geometry, lengths are often specified without reference to the choice of unit length. For example, if a line segment is said to have length 3, it is understood that the segment is 3 times as long as the unit segment. Once a choice of units has been made, two perpendicular line segments having lengths 3 and 4 determine a right triangle with hypotenuse of length 5. The following 3-4-5 right triangle is drawn relative to Jane's units:



Every 3-4-5 right triangle is similar to this one. All have the same shape but they may have different sizes. Here are two examples, one smaller and one larger, but each similar to the 3-4-5 right triangle:



Because these triangles are similar, lengths of corresponding sides have the same ratios. For example,

$$a/4 = b/3 \quad \text{and} \quad a'/4 = b'/3, \quad \text{so} \quad b/a = b'/a' = 3/4.$$

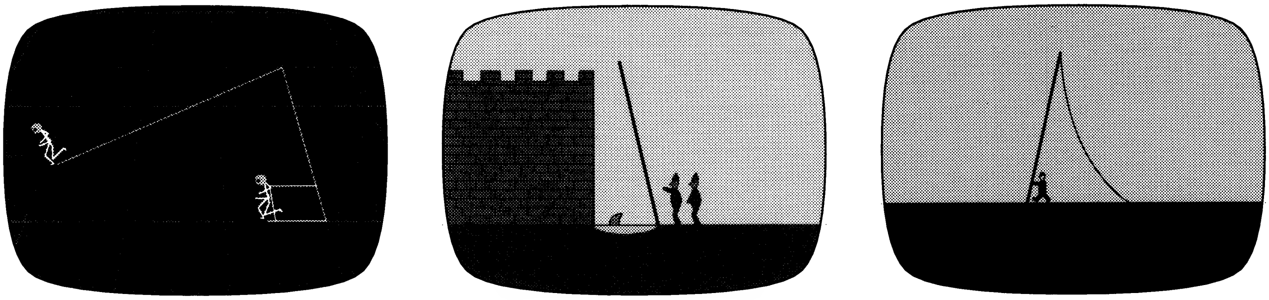
Also,

$$b/3 = c/5 \quad \text{and} \quad b'/3 = c'/5, \quad \text{so} \quad b/c = b'/c' = 3/5.$$

Similarly,

$$a/4 = c/5 \quad \text{and} \quad a'/4 = c'/5, \quad \text{so} \quad a/c = a'/c' = 4/5\ .$$

1. *Three questions from real life*



The video in this capsule shows three real-life situations that lead to the same mathematical problem:

How do you find the length of one side of a right triangle if the lengths of the other two sides are known?

Other real-life examples that lead to the same problem are indicated below. In each case, draw a diagram showing the right triangle involved, and add appropriate labels to indicate which sides are known and which are unknown. You will be asked for solutions to these problems in Section 5.

1. A Greek fisherman on a sailboat needs to attach a metal cable from the top of his mast to the end of the boom. The mast is 8 meters above the boom and the end of the boom is 3 meters in front of the mast. How much cable does he need to make the actual connection?

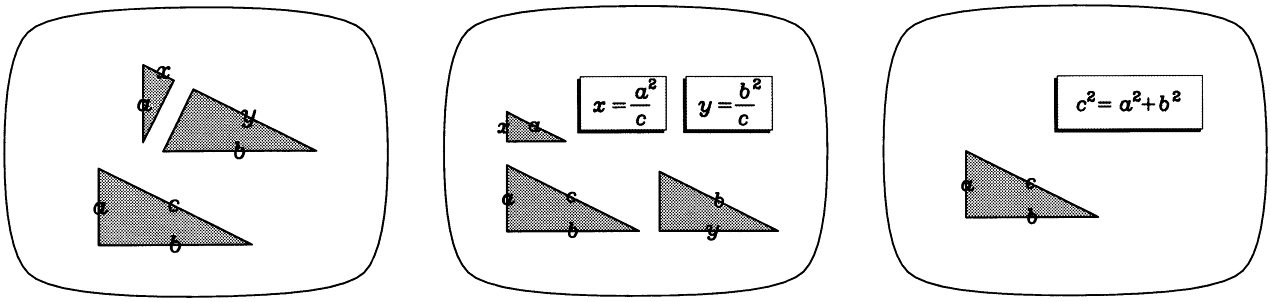
2. A rowboat is crossing a river that is 20 meters wide. The current carries it 25 meters downstream. Assuming the boat moves in a straight line, how far does it actually travel while crossing the stream?

3. A 15-ft ladder is placed against a wall. A path 6 ft wide runs parallel to the wall with its nearest edge 4 ft out from the wall. If the bottom of the ladder is placed on the path, what are the minimum and maximum heights on the wall that can be reached by the ladder?

4. A 20-ft ladder stands vertically against a wall. The lower end is pulled a distance of 2 ft away from the wall. How far down does the top of the ladder move?

5. Give your own example of a real-life situation that leads to the same type of problem.

2. Discovering the Theorem of Pythagoras



1. The derivation in the videotape uses the fact that the three right triangles in Figure 2.1 are similar. (The three right triangles are shown separated in Figure 2.2.) If the acute angles are labeled t and t', s and s', as shown in Figure 2.1, explain why $t = t'$ and $s = s'$.

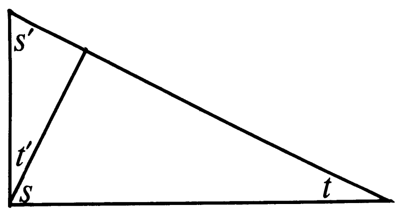


Figure 2.1

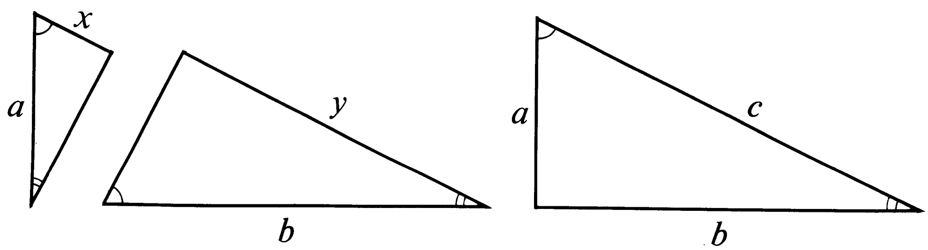


Figure 2.2

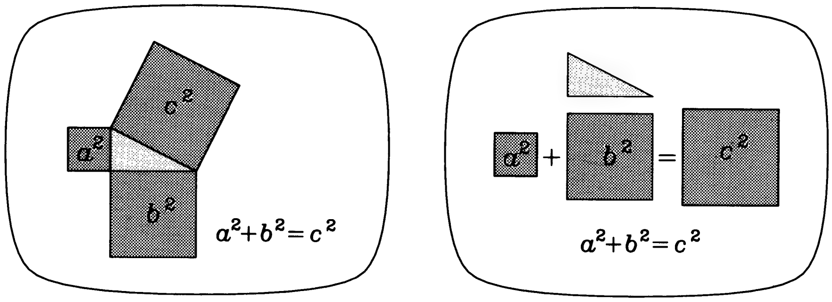
2. In the similar triangles of Figure 2.2, the relations $x/a = a/c$ and $y/b = b/c$, obtained by equating ratios of lengths of corresponding sides, can be cleared of fractions and written as follows:

$$cx = a^2 \quad \text{and} \quad cy = b^2.$$

Draw appropriate rectangles and squares to illustrate the geometric meaning of these equations in terms of areas.

3. In the Pythagorean equation $c^2 = a^2 + b^2$, the right-hand side is symmetric in a and b; that is, its value is not changed if a and b are interchanged. Explain the geometric significance of this symmetry.

3. *Geometric interpretation*



1. Refer to Exercise 2 in Section 2. Add the two equations

$$cx = a^2 \quad \text{and} \quad cy = b^2$$

and obtain an alternate proof of the Theorem of Pythagoras. Make a diagram that shows the geometric meaning of this proof in terms of areas.

4. *Pythagoras*

Historical evidence suggests that the Greek tradition of using logical reasoning to deduce properties of geometric figures was begun during the first half of the sixth century B.C. by Thales of Miletus in Asia Minor. Pythagoras was born about 572 B.C. on the Aegean island of Samos, not far from Miletus, and quite possibly studied under Thales. It appears that Pythagoras, like Thales, visited Egypt and Babylon and traveled extensively in the Orient. When he returned home he found Samos under the tyranny of Polycrates and decided to migrate to the Greek seaport of

Crotona in Southern Italy. There he founded the famous Pythagorean school, a closely-knit brotherhood with secret rites and observances, devoted to the study of philosophy, mathematics, and natural science. As the school prospered, centers of the brotherhood developed in other parts of the Mediterannean world, with the number of disciples and auditors numbering several hundred.

Pythagoras and his followers were zealous in matters of statecraft, advocating aristocratic government and presenting directives and laws to many cities. When they began to have a powerful influence on magistrates, political groups in southern Italy destroyed the buildings of the school and caused the society to disperse. According to one account, Pythagoras fled to Metapontum where he died sometime later, perhaps murdered, at the advanced age of 75 to 80. The brotherhood, although scattered and persecuted, continued to exist for at least two centuries more. To preserve secrecy, all teaching and communication in the Pythagorean brotherhood was oral. Because the disciples attributed all their findings to their revered founder, it is impossible to know which important discoveries were made by Pythagoras himself, and which by other members of the society.

Among the many contributions attributed to the Pythagoreans, some of the most notable are the introduction of: (1) axioms, definitions, and proofs in mathematics; (2) properties of parallel lines and their use in proving that the sum of the angles in a triangle is a straight angle; (3) equality of ratios of corresponding sides of similar triangles; (4) the Pythagorean theorem relating the square of the hypotenuse of a right triangle to the sum of the squares of the legs; (5) the idea that the earth is a globe; (6) the relation of numbers to musical harmony; (7) the relation of numbers to geometry.

The Pythagoreans believed that everything in the universe was related to integers (whole numbers) or to ratios of integers. In particular, they believed that heavenly bodies emit sound based on their size, distance, density, and movement. If they could determine these numbers or ratios, everything would come together in one great harmony--the music of the spheres. Legend has it that the Pythagoreans were shocked when they discovered that the diagonal of a square of side 1 was not the ratio of two integers, and tried to keep this discovery secret. In modern terminology, this result states that the square root of 2 is an irrational number (not the ratio of two integers).

Suggested projects:

1. Consult encyclopedias and other reference materials in your library, and prepare an essay on Pythagoras and the Pythagorean brotherhood. Describe their contributions to musical harmony.

2. To prove that $\sqrt{2}$ is irrational, the Pythagoreans probably assumed that there are integers a and b such that $\sqrt{2} = a/b$ and derived a contradiction. In his book *A Mathematician's Apology*, G. H. Hardy refers to this proof as one of the earliest examples of first-rate mathematics, "as fresh and significant as when it was discovered." Try to construct such a proof by yourself.

5. *Applying the Theorem of Pythagoras*

1. Refer to the real-life situations described in the exercises of Section 1. Use the Pythagorean theorem to obtain numerical answers to each of those problems.

2. Use the Pythagorean theorem to show that in an isosceles right triangle, shown in Figure 5.1, the length of the hypotenuse is $\sqrt{2}$ times the length of each leg.

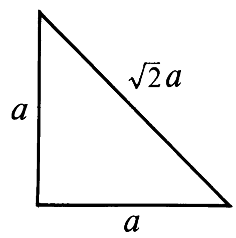


Figure 5.1 Isosceles right triangle.

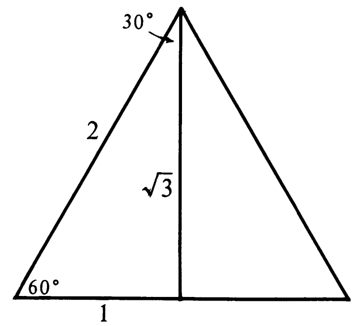


Figure 5.2 A bisected equilateral triangle.

3. If an equilateral triangle is bisected into two right triangles as shown in Figure 5.2, each is called a 30^o-60^o right triangle. Prove that in a 30^o-60^o right triangle the hypotenuse is twice as long as the shorter leg, and the longer leg is $\sqrt{3}$ times as long as the shorter.

4. (a) Start with the isosceles triangle on the right in Figure 5.3, and draw the other right triangles in succession, using only straightedge and compass. Determine the length of each hypotenuse. How many such triangles can be drawn without overlapping the first one?

(b) How would you prove the following statement? If a line of unit length is given, then for each positive integer n a line of length $\sqrt{n}$ can be constructed with straightedge and compass.

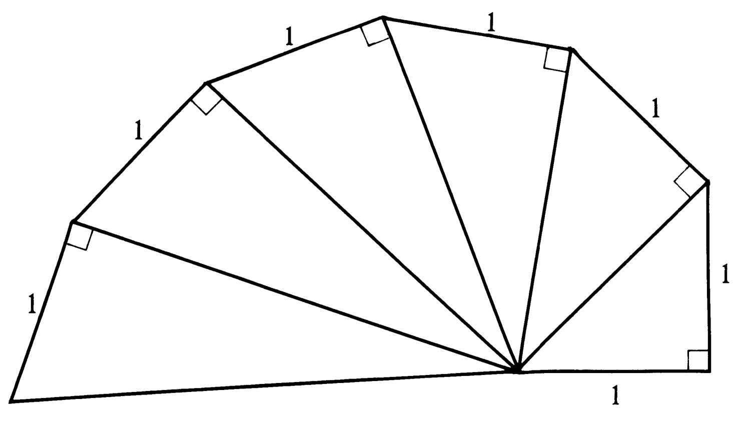


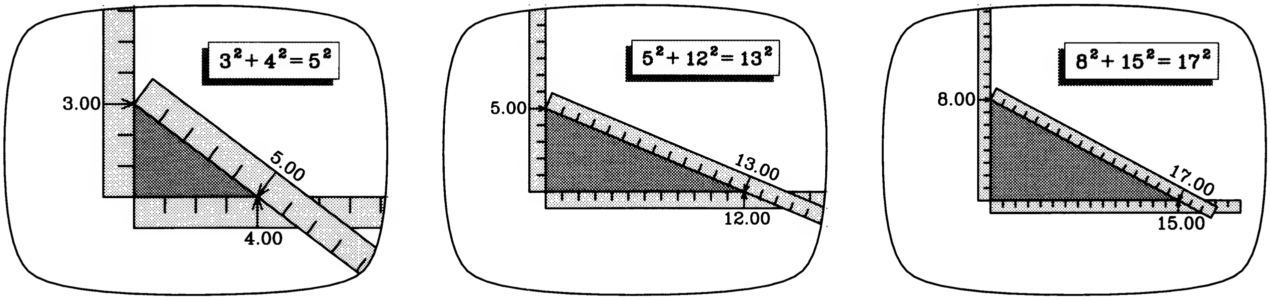
Figure 5.3

5. A rectangle with sides a and b is inscribed in a circle. Find a formula for the determining the radius of the circle in terms of a and b.

6. Calculate the area of a right triangle with legs a, b and hypotenuse 7, if $a + b = \sqrt{59}$.

7. A quarter of a circular disk of radius r is cut out of paper and rolled to form a right circular cone. Show that the altitude of the cone is $\sqrt{15}/4$ times r (nearly 97% of the radius r). What is the relation between altitude and radius if the cone is made from a semicircular disk?

6. *Pythagorean triples*



1. A Pythagorean triple consists of three positive integers a, b, c such that $a^2 + b^2 = c^2$. Find all Pythagorean triples with $a = 20$. (*Suggestion:* Factor both sides of the equation $c^2 - b^2 = 400$.)

2. (a) Find all Pythagorean triples with hypotenuse ≤ 50. (There are 20 such triples altogether.) You may use a hand calculator or a computer if you have one to help with the arithmetic.

(b) How many dissimilar triangles are represented in this list?

(c) The list suggests that all integers, starting with 3, are members of some Pythagorean triple. Do you see why? Can you prove it?

3. In Book X of the *Elements* (in Lemma 1 for Proposition 28), Euclid gave a method for obtaining all Pythagorean triples, although he gave no proof that his method did, indeed, give them all. The method can be summarized by the formulas

$$a = t(m^2 - n^2), \qquad b = 2tmn, \qquad c = t(m^2 + n^2),$$

where t, m and n are arbitrary positive integers with $m > n$.

(a) Verify that these formulas imply $a^2 + b^2 = c^2$.

(b) Determine values of t, m and n that give the following Pythagorean triples:

(3,4,5), (5,12,13), (7,24,25), (8,15,17), (12,35,37), (20,21,29), (20,99,101), (18,80,82), (39,80,89).

4. Prove that every Pythagorean triple a, b, c with $a^2 + b^2 = c^2$ has the following properties:

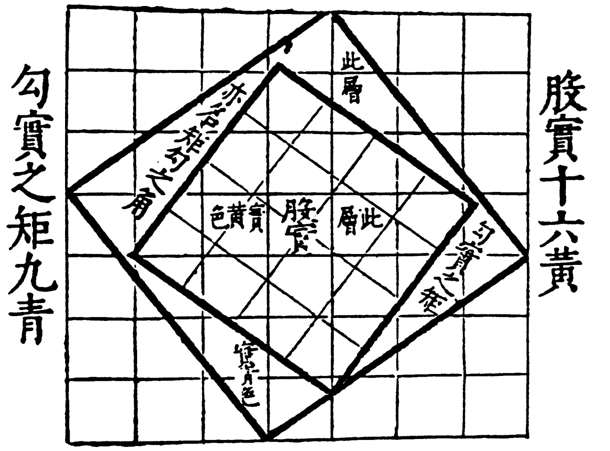
(a) At least one of a or b is divisible by 4.

(b) At least one of a or b is divisible by 3.

(c) At least one of a, b, or c is divisible by 5.

Hint: For part (a), use the formulas in Exercise 3. For part (b), note that if an integer x is not divisible by 3, then $x = 3k \pm 1$ for some integer k. For part (c), note that if an integer x is not divisible by 5, then $x = 5k \pm 1$ or $x = 5k \pm 2$ for some integer k.

7. *The Chinese proof*



1. This exercise describes an algebraic version of the Chinese proof in the video. Start with a square of edge $a + b$ and cut off four right triangles at the corners, each with hypotenuse c, and legs a and b, as shown in Figure 7.1.

(a) Prove that the inner figure is a square.

(b) The area of the inner square, c^2, plus 4 times the area of each triangle, $ab/2$, is equal to the area of the larger square, $(a + b)^2$. Show that this equality of areas implies the Pythagorean theorem.

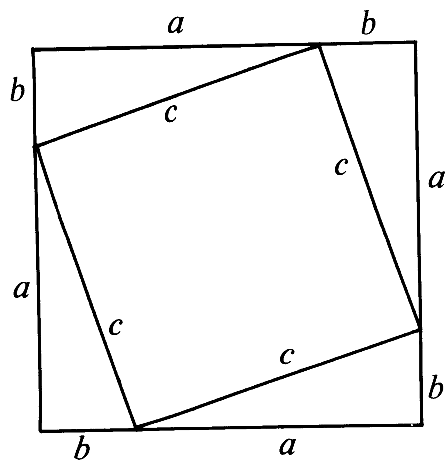


Figure 7.1. Algebraic version of the Chinese proof.

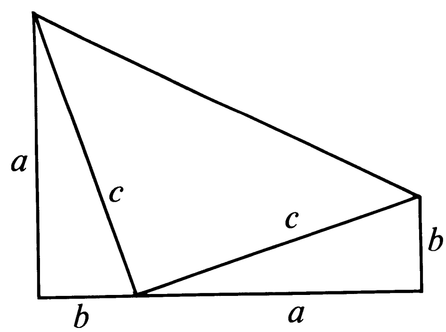


Figure 7.2. Diagram in Garfield's proof.

Note. Essentially the same proof was discovered by U. S. Congressman James A. Garfield, a few years before he became the 20th president of the United States. Garfield used the diagram in Figure 7.2, which is obtained by bisecting that in Figure 7.1.

The diagram in Figure 7.3 below appears in an ancient Chinese text, the *Chou pei suan ching* (*The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*). This text on astronomy and mathematics, written about 1100 B.C., describes a proof of the Pythagorean theorem for the Pythagorean triple (3,4,5). However, the method used is completely general and works for any right triangle.

2. Refer to Figure 7.4. If the legs are a and b and the hypotenuse is c, the small square in the center has area $(b - a)^2$ and the large square has area c^2. Give an algebraic proof of the Pythagorean theorem by adding the areas of the 4 right triangles to that of the small square.

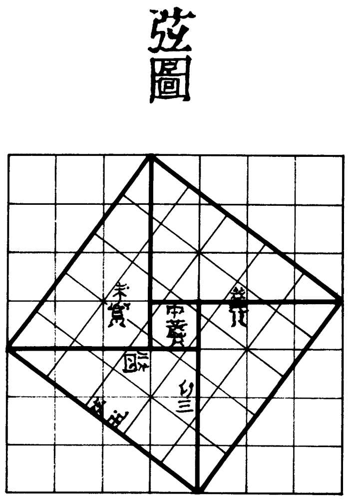


Figure 7.3

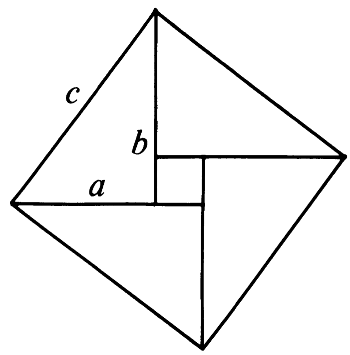


Figure 7.4

3. The Hindu mathematician Bhāskara gave another proof (ca. 1150 A.D.) based on the same diagram. He rearranged the pieces in Figure 7.4 as shown in Figure 7.5, but offered no further explanation than the word "Behold!" Give a proof based on this figure.

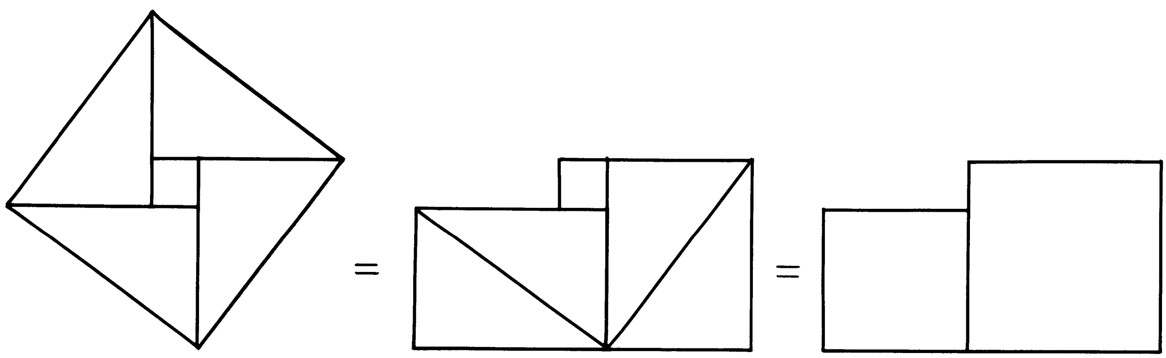


Figure 7.5

8. *Euclid's Elements*

After Alexander the Great died in 323 B.C., his empire was divided among his military leaders and eventually became three empires under separate rule. Egypt was one of these, and Ptolemy became its ruler, naming Alexandria as its capital. Around 306 B.C. he founded the University of Alexandria, the first institution of its kind, a forerunner of modern universities, with lecture rooms, laboratories, and museums. The most impressive part of the university was its great library, which housed more than 600,000 volumes written on rolls of papyrus. This was the largest repository of scholarly works to be found anywhere in that era. The university opened its doors around 300 B.C. and quickly established itself as the intellectual center of the Hellenistic world. Euclid was chosen to head the department of mathematics.

Euclid established his reputation with his *Elements*, a remarkable collection of 13 books that contained much of the mathematics known at that time. Well presented and skillfully arranged in a logical sequence, it was a masterful achievement that had a profound influence on scientific thinking for more than two millenia. Only the Bible has been more widely used or studied.

The first printed edition of Euclid's *Elements* was a Latin translation from Arabic, produced in Venice in 1482. Figure 8.1 shows the elegant opening page of this edition.

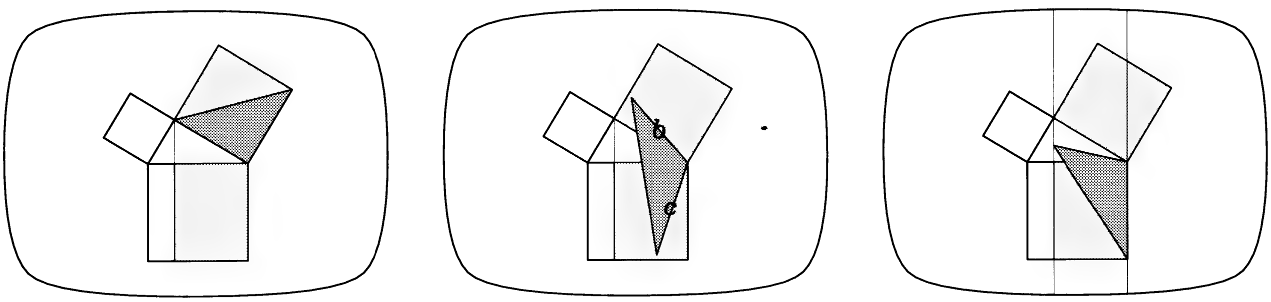
Figure 8.1 The opening page of the first printed edition (1482) of Euclid's *Elements*. Courtesy Huntington Library, San Marino.

Since 1482, more than a thousand editions of Euclid's *Elements* have appeared in print in dozens of languages. No copies on papyrus have been found dating from Euclid's time; all modern editions are based on a revision prepared by Theon of Alexandria seven centuries after Euclid's death. An earlier edition, found in the Vatican Library in the early nineteenth century, together with commentary by early writers, suggests that Theon's version differed only slightly from Euclid's original work.

Books I, II, III and IV are devoted to basic ideas in plane geometry and include many discoveries of the Pythagorean school. Book V describes a theory of proportion by Eudoxus that resolved the logical problems created by the Pythagorean discovery of irrational numbers. This theory eventually provided the foundation for a rigorous treatment of the real number system in the nineteenth century. Book VI applies the theory of proportion to the concept of similarity.

Books VII, VIII and IX deal with properties of the integers (whole numbers) and contain the early beginnings of a branch of mathematics, *The Theory of Numbers*, which has flourished ever since. Book X treats irrational numbers and is considered a remarkable achievement because the results were developed without the aid of modern algebraic notation. Books XI, XII and XIII are concerned with solid geometry. Volumes are treated in Book XII by the method of exhaustion that eventually led to the development of integral calculus twenty centuries later. Most of the geometry appearing in American textbooks is based on Euclid's Books I, III, IV, VI, XI, and XII.

9. *Euclid's proof*



The Pythagorean theorem appears in Euclid's Book I as Proposition 47. Most of the material in Book I was developed by the early Pythagoreans, but the proof of Proposition 47 is attributed to Euclid himself.

1. In the variation of Euclid's proof given in the video, the small square of side a was sheared into a parallelogram, labeled *ABCD* in Figure 9.1 below. Then parallelogram *ABCD* was sheared into a rectangle, one side of which is *AB*, perpendicular to the hypotenuse, and another is *AP,* along the hypotenuse. Use the fact that shearing a parallelogram does not change its area to prove that the length *AB* is equal to c, the length of the hypotenuse. This explains why this rectangle fits properly as part of the large square when it is moved down.

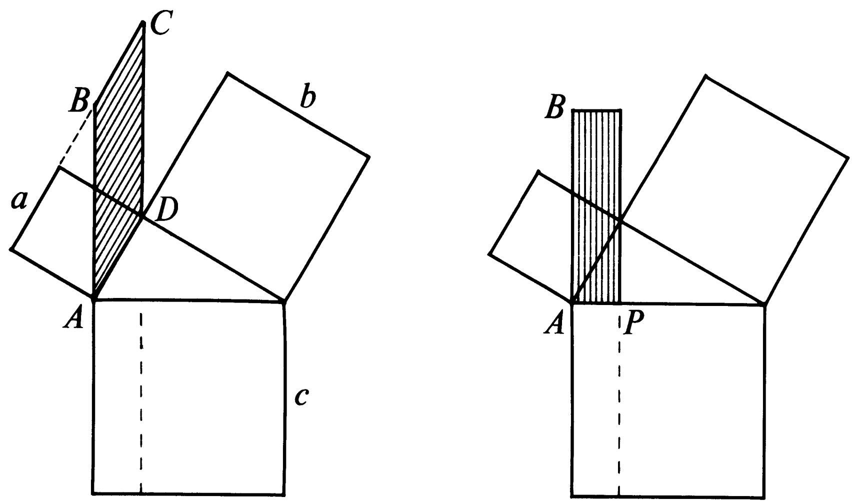
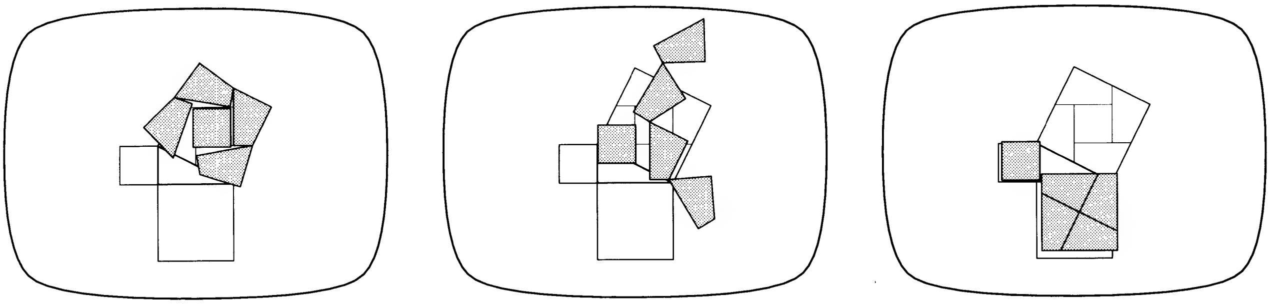


Figure 9.1 Diagram from the variation of Euclid's proof.

2. Euclid's Book I, Proposition 48, contains the converse of the Pythagorean theorem: If a triangle has the property that the sum of the squares of two sides is equal to the square of the third side, then the triangle is a right triangle. Try to construct a proof of Proposition 48.

Note. Carpenters often construct right angles by laying out a triangle whose sides have lengths 3, 4 and 5. First they lay out the two legs of lengths 3 and 4 so they appear approximately perpendicular, then they adjust the angle until the hypotenuse has length exactly equal to 5. The converse of the Pythagorean theorem guarantees that the legs will be perpendicular. Some textbooks assert that the ancient Egyptians used this method in laying out right angles when constructing their temples and pyramids. A papyrus fragment from the twelfth dynasty shows that the Egyptians knew the relation $3^2 + 4^2 = 5^2$, but there is no evidence that the Egyptians knew or could prove the right-angle property of the figure involved.

10. *A dissection proof*



1. The diagram in Figure 10.1 is related to the dissection proof in this portion of the video. The original right triangle has legs a, b and hypotenuse c. The smaller square, of side a, is translated horizontally until its left edge bisects the hypotenuse, then vertically until it is centered on the large square, of side c. Cuts made out to the edges form four quadrilateral pieces. This exercise shows why the four pieces can be rearranged to fill the square of side b exactly.

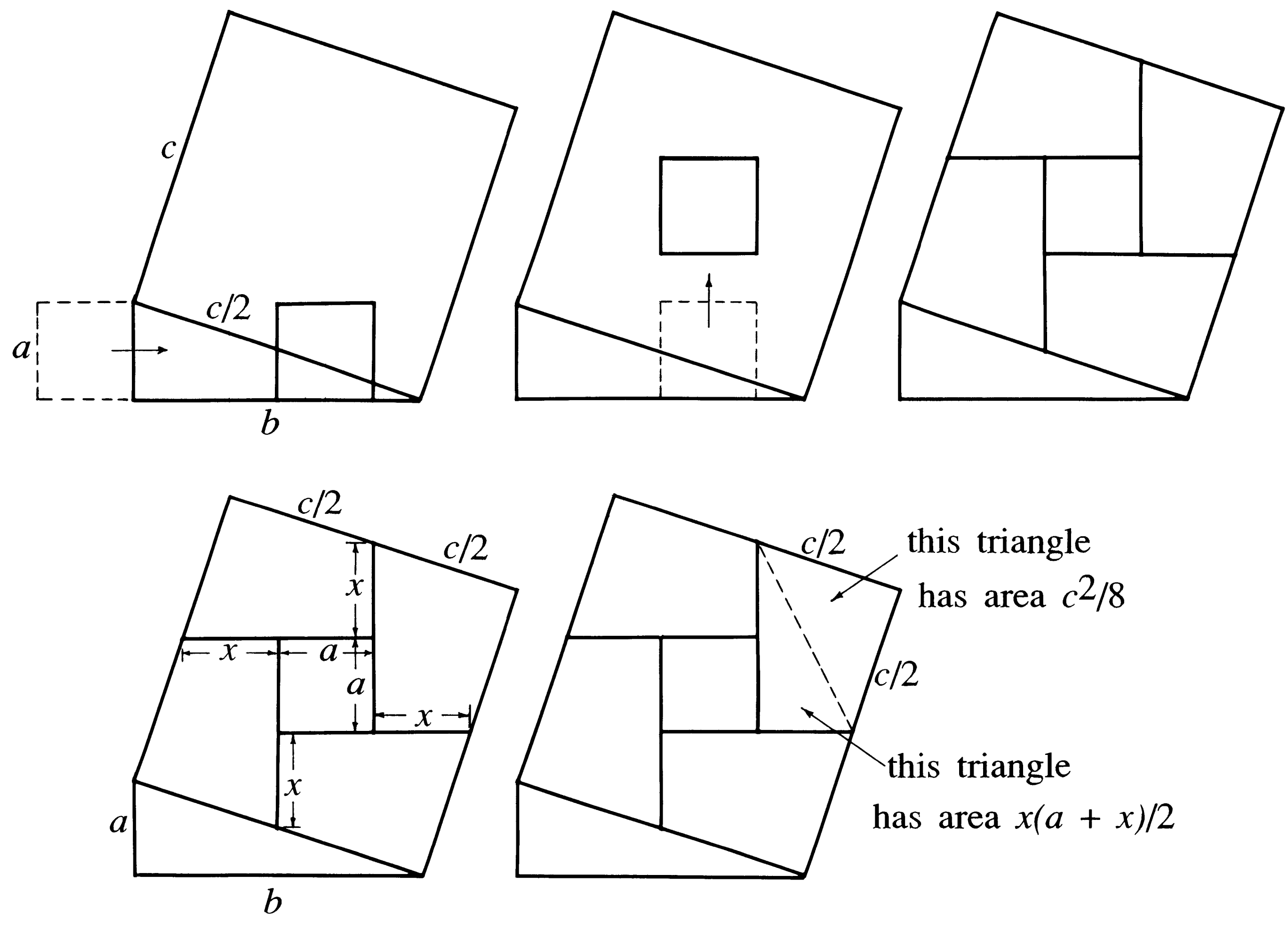


Figure 10.1

(a) Prove that the cuts meet the outside edges of the large square at their centers. This shows that each of the two outer edges of the quadrilateral pieces has length $c/2$.

(b) The inner edges of each quadrilateral piece can be labeled as x and $a + x$, as shown in Figure 10.1. Show that $x = (b - a)/2$ and that $a + x = (b + a)/2$.

(c) Let A denote the area of each quadrilateral piece. Show that $4A = c^2 - a^2$ and also that $4A = 2x(a + x) + c^2/2$. Use these two expressions to obtain $b^2 = c^2 - a^2$.

2. This exercise outlines a method for constructing arbitrarily many dissection proofs of the Pythagorean theorem. It is based on the idea of tiling the plane in two different ways. Begin with a right triangle with legs a, b and hypotenuse c. An example is shown in Figure 10.2.

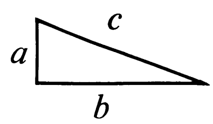


Figure 10.2

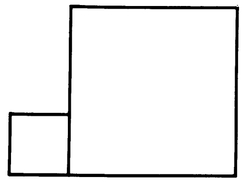


Figure 10.3

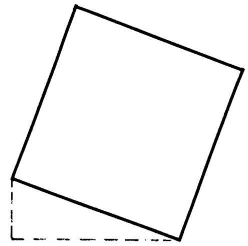


Figure 10.4

Form an L-shaped tile as shown in Figure 10.3, using two adjacent squares of areas a^2 and b^2, and imagine the entire plane covered with these tiles, as in Figure 10.5. Next imagine a second tiling of the plane by squares of area c^2, parallel to the square in Figure 10.4, as shown in Figure 10.6.

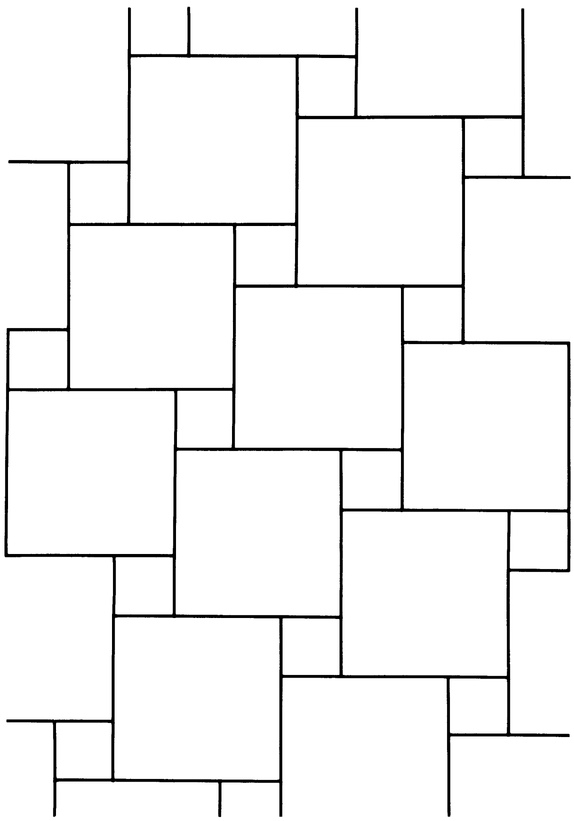


Figure 10.5

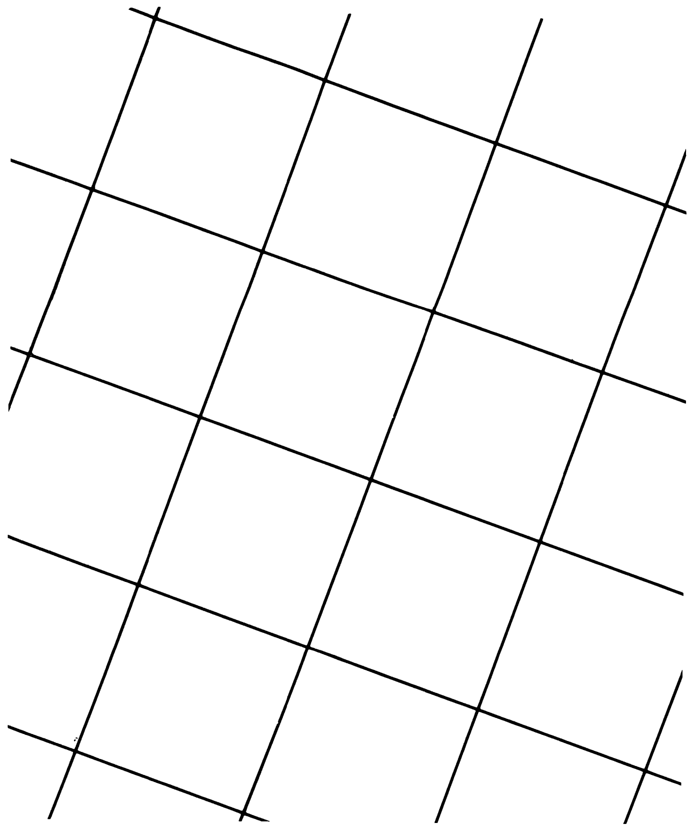


Figure 10.6

Next, the squares of the second tiling can be superimposed over those of the first tiling without changing the orientation. Figures 10.7 and 10.8 illustrate two ways of doing this. In Figure 10.7, the vertices of the second tiling are placed directly in the center of the square whose

area is b^2. This produces a dissection of the large square into a small square of area a^2, plus four congruent quadrilateral pieces of total area b^2, the same dissection that was discussed in Exercise 1. A different dissection is shown in Figure 10.8, and there are infinitely many other possibilities depending on the location of the vertices of the second tiling. No matter where the vertices are located, the resulting dissection of the large square is repeated in each copy of the large square. This is because a displacement along the hypotenuse of a right triangle can also be achieved by a horizontal and a vertical displacement along its legs.

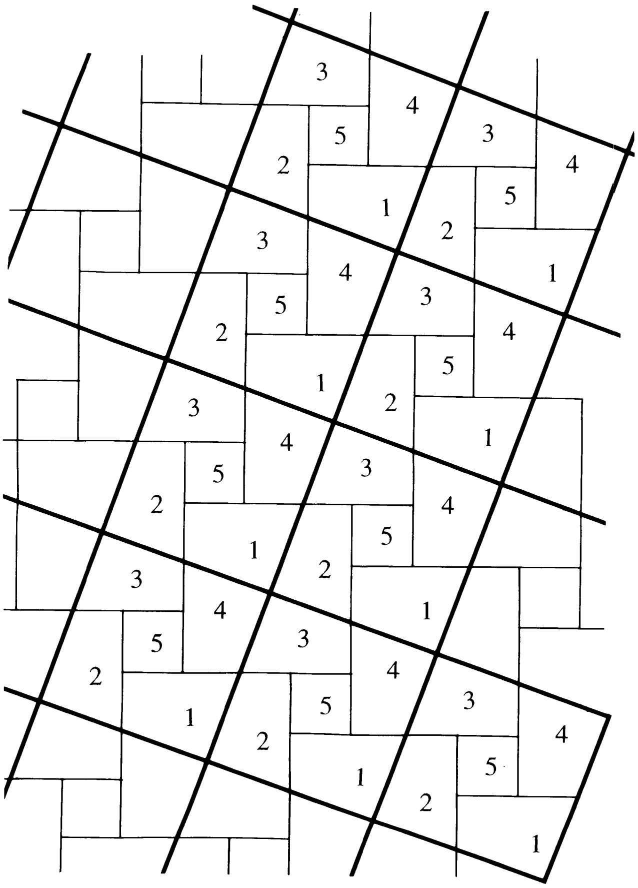


Figure 10.7

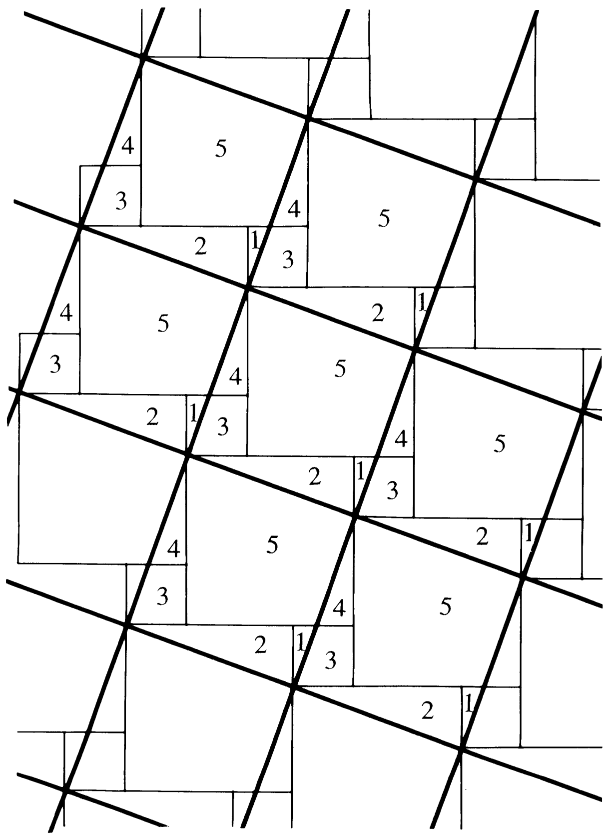


Figure 10.8

3. Leonardo da Vinci is said to have devised a dissection proof based on Figure 10.9. Construct a proof based on this diagram.

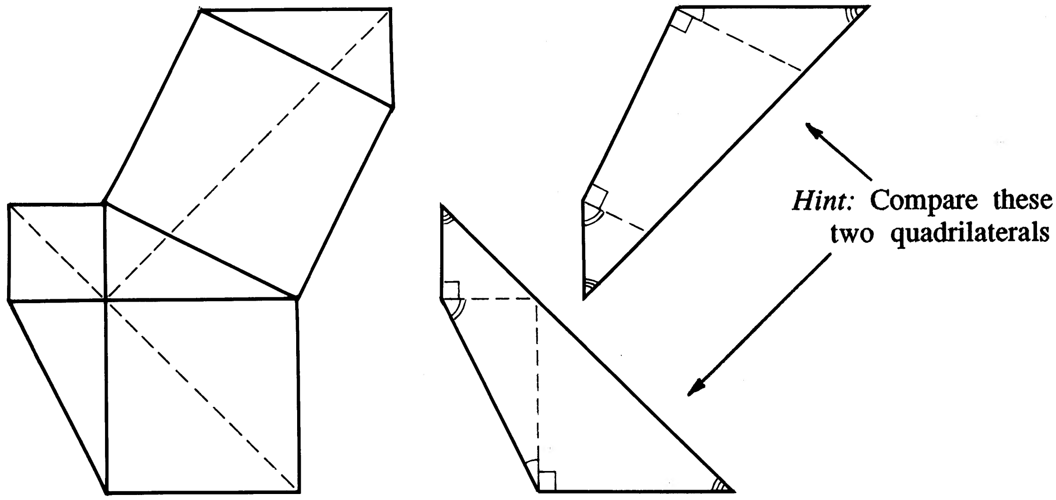
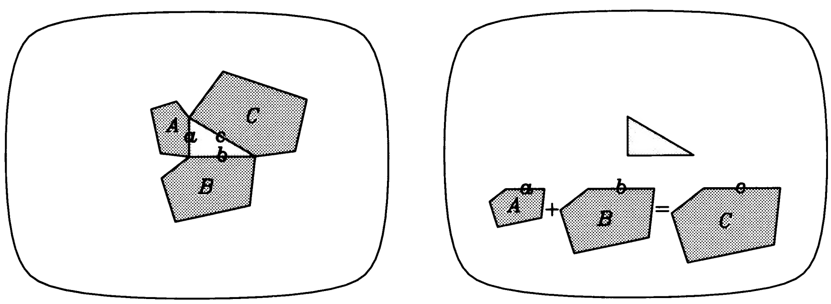


Figure 10.9

11. *Euclid's Book VI, Proposition 31*



1. Given two equilateral triangles with areas A and B, respectively. Using only straightedge and compass, construct a third equilateral triangle whose area is $A + B$.

2. Figure 11.1 shows two similar polygons with corresponding sides on the left having lengths a and b. Using only straightedge and compass, draw a third polygon similar to these whose area is the sum of the areas of the two given polygons.

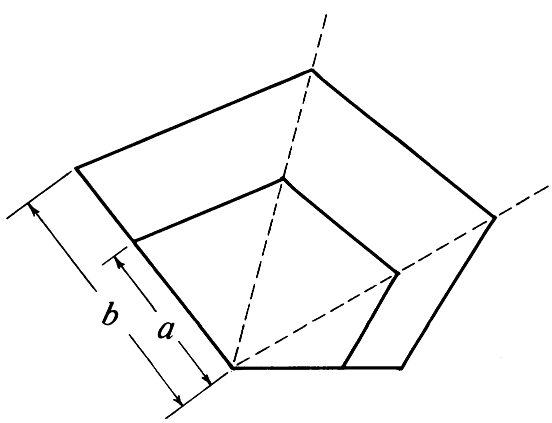
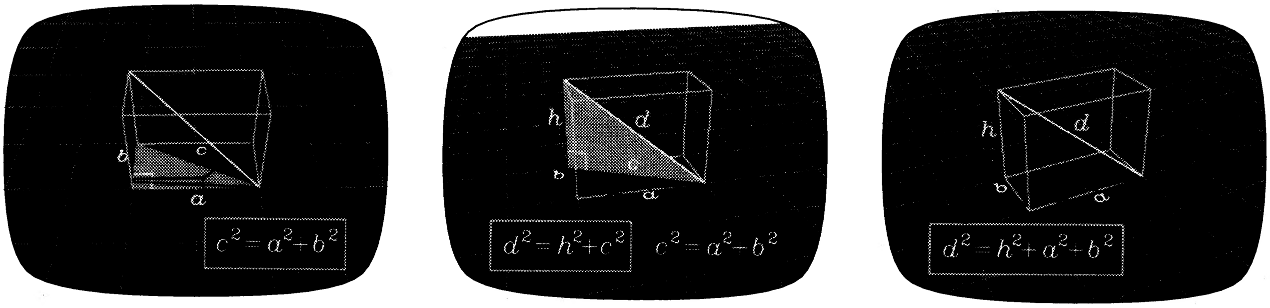


Figure 11.1

13. *The Pythagorean Theorem in 3D*



1. The inside of a carton has length 30 in, width 20 in, and depth 12 in. What is the maximum length of a rod that can be placed in the carton?

2. The length, width and height of a rectangular room are given by $(ab)^2$, $(ac)^2$ and $(bc)^2$, where (a,b,c) is a Pythagorean triple with $a^2 + b^2 = c^2$. Calculate the length of the diagonal from one corner on the floor to the diagonally opposite corner on the ceiling and show that this length is an integer.

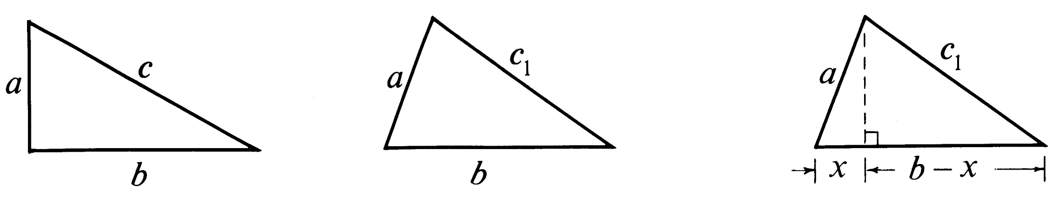
3. According to the three-dimensional version of the Pythagorean theorem, the shortest distance d a fly must travel to get from one corner on the floor of a rectangular room to the diagonally opposite corner on the ceiling is

$$d = \sqrt{a^2 + b^2 + h^2},$$

where a and b are the dimensions of the floor and h is the height of the ceiling above the floor. Find, in terms of a, b, and h, the shortest distance e an ant would travel by walking from one corner to the other. (*Hint:* Fold down the walls and apply the two-dimensional Pythagorean theorem. But note that there is more than one path the ant can take to reach its destination.)

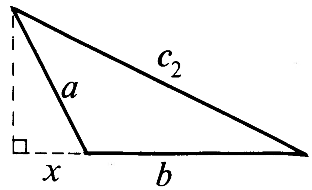
Appendix: *Extending the Pythagorean Theorem*

1. Consider a right triangle with legs a and b and hypotenuse c, shown below in (a). The right angle is decreased so as to form a new triangle with two sides a and b and a third side c_1, as shown in (b). Comparison of (a) and (b) suggests that $c_1 < c$. Prove this as follows: Apply the Pythagorean theorem to each right triangle shown in (c) and deduce that $c_1^2 = a^2 + b^2 - 2bx$, where x is the distance indicated in (c).



(a) (b) (c)

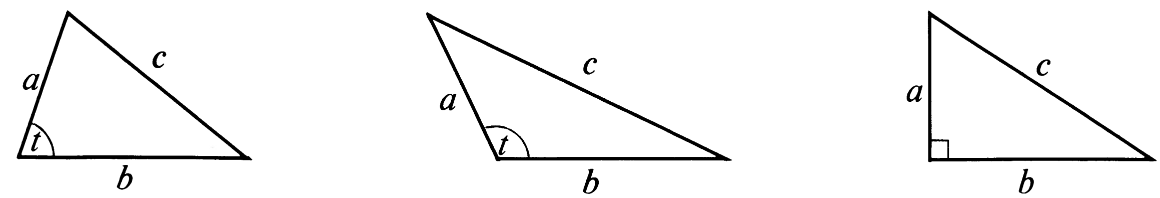
2. By an argument similar to that in Exercise 1, show that $c_2^2 = a^2 + b^2 + 2bx$ in this figure:



Note: The two formulas in Exercises 1 and 2 can be written as a single formula called the *law of cosines,* which relates the lengths of the sides of any triangle. If a triangle has sides of lengths a, b and c, then

$$c^2 = a^2 + b^2 - 2ab \cos t, \qquad \text{(law of cosines)}$$

where t is the angle between the two sides of lengths a and b. If the angle t is smaller than a right angle, as in Exercise 1, $\cos t$ is the ratio x/a. If the angle t is greater than a right angle, as in Exercise 2, $\cos t$ is the negative of the ratio x/a. In a right triangle with legs a and b, the angle t is a right angle, $\cos t = 0$, and the law of cosines becomes the Theorem of Pythagoras.



$c^2 = a^2 + b^2 - 2bx$ $\qquad$ $c^2 = a^2 + b^2 + 2bx$ $\qquad$ $c^2 = a^2 + b^2$

All three cases are described by the law of cosines: $c^2 = a^2 + b^2 - 2ab \cos t$.

Suggested references for further study

1. Asger Aaboe, *Episodes from the Early History of Mathematics,* New Mathematical Library, Random House, 1964.

2. G. D. Chakerian, C. D. Crabill, and S. K. Stein, *Geometry--A Guided Inquiry,* Sunburst Communications, Inc., 1987.

3. R. Courant and H. Robbins, *What is Mathematics?*, Oxford University Press, 1978.

4. H. S. M. Coxeter, *Introduction to Geometry,* John Wiley & Sons, 1969.

5. Howard Eves, *An Introduction to the History of Mathematics,* Holt, Rinehart and Winston, 1964.

6. B. L. van der Waerden, *Geometry and Algebra in Ancient Civilizations,* Springer-Verlag, 1983.

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